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A Fourier Analysis of Classical and Fractional Diffusion: Smoothing, Decay, and Comparison

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ABSTRACT

This paper investigates the classical heat equation and the fractional diffusion equation using Fourier transform methods. By transforming both equations into the frequency domain, explicit representations of their solutions are obtained. These representations are then used to analyze several analytical properties, including smoothing effects, decay estimates, and the limiting behavior of the fractional model as the fractional order approaches one. The results show that the classical heat equation exhibits stronger smoothing and faster decay, while the fractional diffusion equation produces heavier spatial tails due to the intrinsic nonlocal behavior of the fractional Laplacian. This study highlights the main differences and connections between classical and fractional diffusion processes within a unified Fourier analysis framework.

Keywords: heat equation, fractional diffusion equation, Fourier transform, fractional Laplacian

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INTRODUCTION

The heat equation represents one of the central models in the theory of partial differential equations, describing diffusion processes such as heat propagation and particle transport. In its classical form,

$$\partial_t u - \Delta u = 0, \qquad u(x,0) = u_0(x)$$

its analytical properties are well established; see (Evans, 2010; Friedman, 2008; Pazy, 1983). The application of the Fourier transform (FT) yields an explicit representation of the solution

$$\hat{u}(\xi,t) = e^{-|\xi|^2 t}\hat{u}_0(\xi)$$

which reveals smoothing effects, decay behavior, and semigroup structure.

Fractional diffusion equations generalize this model by replacing the Laplacian with the fractional Laplacian:

$$\partial t u + (-\Delta)\alpha u = 0, \qquad 0 < \alpha < 1$$

The fractional Laplacian has been extensively studied; see (Caffarelli & Silvestre, 2007; Di Nezza et al., 2012; Lischke et al., 2020). Further developments on fractional operators and nonlocal diffusion can be found in (Acosta & Borthagaray, 2017; D'Elia &

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Gunzburger, 2013; Pozrikidis, 2016; Ros-Oton & Serra, 2014; Sato, 1999; Valdinoci, 2009). From a probabilistic perspective, anomalous diffusion and Lévy processes have been investigated by (Applebaum, 2009; Meerschaert & Sikorskii, 2012; Metzler & Klafter, 2000)

Recent studies also address analytical and numerical aspects of fractional diffusion equations, including finite-volume schemes and inverse problems; see, for example, (Bailo et al., 2025; Bonito & Pasciak, 2015; Kossowski & Przeradzki, 2024; Mamanazarov & Suragan, 2024).

Despite these developments, most existing works treat classical and fractional diffusion models separately. A direct analytical comparison within a unified Fourier framework remains limited.

This work primarily aims to establish a unified Fourier analytic framework. We derive explicit Fourier representations and use them to analyze smoothing effects, decay estimates, and the limiting behavior as $\alpha \to 1$. This approach highlights, in a systematic way, the differences between Gaussian diffusion and nonlocal anomalous diffusion through their Fourier multipliers.

The organization of this paper is as follows. Section 2 presents the materials and methods, including the necessary preliminaries. Section 3 provides the results and discussion, where the Fourier representations are derived and the smoothing and decay properties are analyzed. Finally, Section 4 concludes the paper.

MATERIAL AND METHODS

This section presents several essential definitions and properties that will be utilized throughout the paper. The material presented here follows standard references in partial differential equations, Fourier analysis, and fractional operators; see (Caffarelli & Silvestre, 2007; Di Nezza et al., 2012; Evans, 2010; Grafakos, 2014; Lischke et al., 2020; Pazy, 1983; Pozrikidis, 2016).

Definition 1. (Fourier Transform) Let $f \in L^1(\mathbb{R}^n)$. The FT of a function f is defined as follows

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)\, e^{-ix\cdot\xi}\, dx, \tag{1}$$

for $\xi \in \mathbb{R}^n$. The original function can be recovered via the inverse FT

$$f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(\xi) e^{ix\cdot\xi}\, d\xi.$$

The FT extends to $L^2(\mathbb{R}^n)$ and becomes a unitary operator.

Theorem 1. (Plancherel Identity (Evans, 2010; Friedman, 2008)**)** Let $f \in L^2(\mathbb{R}^n)$. Then

$$\| f \|_{L^2(\mathbb{R}^n)} = \| \hat{f} \|_{L^2(\mathbb{R}^n)}. \tag{2}$$

A key property of the Fourier transform in the context of partial differential equations is that it transforms differential operators into multiplication operators in the frequency domain. In particular,

$$\widehat{(\Delta f)}(\xi) = -|\xi|^2 \hat{f}(\xi). \tag{3}$$

Definition 2. (Fractional Laplacian (Caffarelli & Silvestre, 2007; Di Nezza et al., 2012; Lischke et al., 2020)**)** Let $0 < \alpha < 1$. The fractional Laplacian $(-\Delta)^\alpha$ is defined in the

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Fourier domain by

$$\widehat{(-\Delta)^{\alpha}}(\xi) = |\xi|^{2\alpha} \hat{f}(\xi). \tag{4}$$

Equivalently, the fractional Laplacian admits the singular integral representation

$$(-\Delta)^{\alpha} f(x) = C_{n,\alpha} \, \mathrm{P.V.} \int_{\mathbb{R}^n} \frac{f(x) - f(y)}{|x-y|^{n+2\alpha}} \, dy,$$

where P.V. indicates the Cauchy principal value, while $C_{n,\alpha}$denotes a normalization constant determined by the dimension $n$and the parameter α.

Theorem 2. (Heat Semigroup (Pazy, 1983)**)** Let $f \in L^2(\mathbb{R}^n)$. The classical heat equation generates the semigroup

$$S(t) = e^{t\Delta}, \tag{5}$$

which can be expressed in terms of its FT as

$$\widehat{S(t)f}(\xi) = e^{-|\xi|^2 t} \hat{f}(\xi). \tag{6}$$

Similarly, the fractional diffusion equation generates the fractional semigroup

$$S_{\alpha}(t) = e^{-t(-\Delta)^{\alpha}},$$

which satisfies

$$\widehat{S_{\alpha}(t)f}(\xi) = e^{-|\xi|^{2\alpha} t} \hat{f}(\xi).$$

Both operators satisfy the semigroup property

$$S(t+s) = S(t)S(s), S_{\alpha}(t+s) = S_{\alpha}(t)S_{\alpha}(s).$$

The findings of this section are utilized in the subsequent study of the classical heat equation and the fractional diffusion equation.

RESULTS AND DISCUSSION

Fourier Representation of the Solutions

This subsection presents the derivation of explicit solution representations for the classical heat equation and the fractional diffusion equation through the use of the FT. The FT is a commonly employed method for addressing linear partial differential equations.

1. Classical Heat Equation

The Cauchy problem for the classical heat equation is formulated as follows:

$$\partial_t u - \Delta u = 0, \qquad u(x,0) = u_0(x) \tag{7}$$

Lemma 1. Suppose that $u(x,t)$is a sufficiently smooth solution to equation (7). By performing the FT with respect to the spatial variable x, we obtain:

$$\partial_t \hat{u}(\xi,t) + |\xi|^2 \hat{u}(\xi,t) = 0 \tag{8}$$

Proof. Taking theFT, defined in (1), in the spatial variable x for equation (7) yields

$$\widehat{\partial_t u}(\xi,t) - \widehat{\Delta u}(\xi,t) = 0.$$

Since the FT acts only on the spatial variable, it commutes with differentiation in time. Hence,

$$\widehat{\partial_t u}(\xi,t) = \partial_t \hat{u}(\xi,t).$$

On the other hand, using the property of the FT of the Laplacian given in (3), we have

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$$\widehat{\Delta u}(\xi,t) = -|\ \xi\ |^2\ \hat{u}(\xi,t).$$

Substituting these identities into the transformed equation yields

$$\partial_t\hat{u}(\xi,t) - (-|\ \xi\ |^2)\hat{u}(\xi,t) = 0.$$

Therefore, we obtain the ordinary differential equation

$$\partial_t\hat{u}(\xi,t) + |\xi|^2\hat{u}(\xi,t) = 0,$$

which proves (8).

Theorem 3. Let $u_0 \in L^2(\mathbb{R}^n)$. The solution of the heat equation (7) is given in the Fourier domain by

$$\hat{u}(\xi,t) = e^{-|\xi|^2 t}\hat{u}_0(\xi). \tag{9}$$

Proof. From Lemma (8), the FT $\hat{u}(\xi,t)$ satisfies the ordinary differential equation

$$\partial_t\hat{u}(\xi,t)+|\ \xi\ |^2\ \hat{u}(\xi,t) = 0.$$

For each fixed $\xi \in \mathbb{R}^n$, this is a first-order linear ordinary differential equation in the variable t. Its general solution is given by

$$\hat{u}(\xi,t) = C(\xi)e^{-|\xi|^2 t},$$

where $C(\xi)$ is an arbitrary function determined only by ξ.

To determine $C(\xi)$, we use the initial condition $u(x,0) = u_0(x)$. Taking the FT with respect to x, we obtain

$$\hat{u}(\xi,0) = \hat{u}_0(\xi),$$

substituting $t = 0$ into the general solution yields

$$C(\xi) = \hat{u}_0(\xi).$$

Therefore,

$$\hat{u}(\xi,t) = e^{-|\xi|^2 t}\hat{u}_0(\xi),$$

which proves (9).

Applying the inverse FT defined in (1), the solution can be written as

$$u(x,t) = G_t * u_0(x), \tag{10}$$

where G_t is the classical Gaussian heat kernel.

2. Fractional Diffusion Equation

Next we consider the fractional diffusion equation

$$\partial_t u + (-\Delta)^\alpha u = 0, \qquad 0 < \alpha < 1, \tag{11}$$

with initial condition $u(x,0) = u_0(x)$.

Lemma 2. Let $u(x,t)$ be a sufficiently smooth solution of (11). Taking the Fourier transform with respect to the spatial variable x, one obtains

$$\partial_t\hat{u}(\xi,t)+|\ \xi\ |^{2\alpha}\ \hat{u}(\xi,t) = 0. \tag{12}$$

Proof. Applying the FT defined in (1) with respect to the spatial variable x to equation (11), we obtain

$$\widehat{\partial_t u}(\xi,t) + \widehat{(-\Delta)^\alpha} u(\xi,t) = 0.$$

Since the FT acts only on the spatial variable, it commutes with differentiation in time, so that

$$\widehat{\partial_t u}(\xi,t) = \partial_t\hat{u}(\xi,t).$$

On the other hand, by the definition of the fractional Laplacian in the Fourier domain (4), we have

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$$\widehat{(-\Delta)^\alpha u}(\xi,t) = |\xi|^{2\alpha}\,\hat{u}(\xi,t).$$

Substituting these expressions into the transformed equation yields

$$\partial_t \hat{u}(\xi,t) + |\xi|^{2\alpha}\,\hat{u}(\xi,t) = 0.$$

which proves (12).

Theorem 4. Let $u_0 \in L^2(\mathbb{R}^n)$. The solution of the fractional diffusion equation (11) admits the Fourier representation

$$\hat{u}(\xi,t) = e^{-|\xi|^{2\alpha}t}\hat{u}_0(\xi). \tag{13}$$

Proof. From Lemma (12), the FT $\hat{u}(\xi,t)$ satisfies the ordinary differential equation

$$\partial_t \hat{u}(\xi,t) + |\xi|^{2\alpha}\,\hat{u}(\xi,t) = 0. \tag{14}$$

For each fixed $\xi \in \mathbb{R}^n$, this is a first-order linear ordinary differential equation in t. Its general solution is given by

$$\hat{u}(\xi,t) = C(\xi)e^{-|\xi|^{2\alpha}t},$$

where $C(\xi)$ depends only on ξ.

Using the initial condition $u(x,0) = u_0(x)$ and taking the FT, we obtain

$$\hat{u}(\xi,0) = \hat{u}_0(\xi).$$

Substituting $t = 0$ into the general solution yields

$$C(\xi) = \hat{u}_0(\xi).$$

Hence,

$$\hat{u}(\xi,t) = e^{-|\xi|^{2\alpha}t}\hat{u}_0(\xi),$$

which proves (13).

This representation shows that the classical heat equation and the fractional diffusion equation have similar Fourier structures, with the difference that the classical Laplacian multiplier $|\xi|^2$ is replaced by $|\xi|^{2\alpha}$ in the fractional case.

Analytical Properties of the Solutions

In this section, we investigate several analytical properties of the solutions derived from the Fourier representations. In particular, we study smoothing effects and decay estimates for both the classical heat equation and the fractional diffusion equation. These properties follow from the Fourier representations (9) and (13).Related analytical properties for nonlocal and fractional diffusion models have also been discussed in (Acosta & Borthagaray, 2017; Bonito & Pasciak, 2015; Ros-Oton & Serra, 2014).

1. **Smoothing Effects**

The heat equation is well known for its strong smoothing effect. Even if the initial data is not smooth, the solution becomes smooth for any positive time.

Theorem 5. (Smoothing Effect for the Heat Equation) Let $u_0 \in L^2(\mathbb{R}^n)$ and let $u(x,t)$ be the solution of the heat equation given by (9). Then for every $t > 0$,

$$u(\cdot,t) \in H^k(\mathbb{R}^n), \tag{15}$$

for all integers $k \geq 0$.

Proof. Using the Fourier representation (9), we have

$$\hat{u}(\xi,t) = \hat{u}_0(\xi)e^{-|\xi|^2 t}. \tag{16}$$

To study the regularity of $u(\cdot,t)$, we consider the Sobolev norm of order k

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$$\| u(\cdot,t) \|_{H^k}^2 = \int_{\mathbb{R}^n} | \hat{u}(\xi,t) |^2 \, (1+| \xi |^2)^k \, d\xi. \tag{17}$$

Substituting (16) into (17), we obtain

$$\| u(\cdot,t) \|_{H^k}^2 = \int_{\mathbb{R}^n} | \hat{u}_0(\xi) |^2 \, (1+| \xi |^2)^k e^{-2|\xi|^2 t} \, d\xi. \tag{18}$$

Now, for each fixed $t > 0$, the weight function in (18) satisfies

$$(1+| \xi |^2)^k e^{-2|\xi|^2 t} \leq C_k(t), \tag{19}$$

for all $\xi \in \mathbb{R}^n$, since the exponential decay dominates any polynomial growth. Using (19) in (18), we obtain

$$\| u(\cdot,t) \|_{H^k}^2 \leq C_k(t) \int_{\mathbb{R}^n} | \hat{u}_0(\xi) |^2 \, d\xi. \tag{20}$$

Finally, by the Plancherel identity (2), estimate (20) yields

$$\| u(\cdot,t) \|_{H^k}^2 \leq C_k(t) \, \| u_0 \|_{L^2}^2 < \infty. \tag{21}$$

Hence $u(\cdot,t) \in H^k(\mathbb{R}^n)$ for all $k \geq 0$ and all $t > 0$, which shows that the heat equation has a strong smoothing effect. Next we analyze the smoothing behavior of the fractional diffusion equation.

Theorem 6. (Smoothing Effect for Fractional Diffusion). Let $u_0 \in L^2(\mathbb{R}^n)$ and let $u(x,t)$ be the solution of the fractional diffusion equation given by (13). Then for every $t > 0$,

$$u(\cdot,t) \in H^{2\alpha}(\mathbb{R}^n). \tag{22}$$

Proof. From the Fourier representation (13), we have

$$\hat{u}(\xi,t) = \hat{u}_0(\xi) e^{-|\xi|^{2\alpha} t}. \tag{23}$$

To analyze the regularity, we consider the Sobolev norm of order 2α

$$\| u(\cdot,t) \|_{H^{2\alpha}}^2 = \int_{\mathbb{R}^n} | \hat{u}(\xi,t) |^2 \, (1+| \xi |^2)^{2\alpha} \, d\xi. \tag{24}$$

Substituting (23) into (24), we obtain

$$\| u(\cdot,t) \|_{H^{2\alpha}}^2 = \int_{\mathbb{R}^n} | \hat{u}_0(\xi) |^2 \, (1+| \xi |^2)^{2\alpha} e^{-2|\xi|^{2\alpha} t} \, d\xi. \tag{25}$$

For each fixed $t > 0$, the weight function in (25) satisfies

$$(1+| \xi |^2)^{2\alpha} e^{-2|\xi|^{2\alpha} t} \leq C_\alpha(t), \tag{26}$$

for all $\xi \in \mathbb{R}^n$, since the exponential decay dominates the polynomial growth. Using (26) in (25), we obtain

$$\| u(\cdot,t) \|_{H^{2\alpha}}^2 \leq C_\alpha(t) \int_{\mathbb{R}^n} | \hat{u}_0(\xi) |^2 \, d\xi. \tag{27}$$

Finally, by the Plancherel identity (2), estimate (27) yields

$$\| u(\cdot,t) \|_{H^{2\alpha}}^2 \leq C_\alpha(t) \, \| u_0 \|_{L^2}^2 < \infty. \tag{28}$$

Hence $u(\cdot,t) \in H^{2\alpha}(\mathbb{R}^n)$ for all $t > 0$.

2. Decay Estimates

Another important property of diffusion equations is the decay of the solution over time.

Lemma 3. Let $u(x,t)$ be the solution of the heat equation with initial data $u_0 \in L^1(\mathbb{R}^n)$. Then

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$$\| u(\cdot,t) \|_{L^\infty} \leq Ct^{-n/2} \| u_0 \|_{L^1}, \tag{29}$$

for all $t > 0$.

Proof. Using the convolution representation (10), we write

$$u(x,t) = \int_{\mathbb{R}^n} G_t\,(x-y)u_0(y)\,dy. \tag{30}$$

Taking absolute values in (30), we obtain

$$| u(x,t) | \leq \int_{\mathbb{R}^n} | G_t(x-y) | \;\; | u_0(y) | \;dy. \tag{31}$$

The Gaussian heat kernel is explicitly given by

$$G_t(x) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}, \tag{32}$$

which implies the bound

$$| G_t(x) | \leq \frac{C}{t^{n/2}}. \tag{33}$$

Substituting (33) into (31), we obtain

$$| u(x,t) | \leq \frac{C}{t^{n/2}} \int_{\mathbb{R}^n} | u_0(y) | \;dy = Ct^{-n/2} \| u_0 \|_{L^1}. \tag{34}$$

Taking the supremum over $x \in \mathbb{R}^n$in (34), we obtain

$$\| u(\cdot,t) \|_{L^\infty} \leq Ct^{-n/2} \| u_0 \|_{L^1}, \tag{35}$$

which proves (29).

Lemma 4. Let $u(x,t)$ be the solution of the fractional diffusion equation with initial data $u_0 \in L^1(\mathbb{R}^n)$. Then

$$\| u(\cdot,t) \|_{L^\infty} \leq Ct^{-\frac{n}{2\alpha}} \| u_0 \|_{L^1}. \tag{36}$$

Proof. Using the Fourier representation (13) and the inverse FT (1), the solution can be written as a convolution with the fractional heat kernel

$$u(x,t) = K_\alpha(t) * u_0(x). \tag{37}$$

Taking absolute values in (37), we obtain

$$| u(x,t) | \leq \int_{\mathbb{R}^n} | K_\alpha(x-y,t) || u_0(y) | \;dy. \tag{38}$$

The fractional heat kernel satisfies the estimate

$$| K_\alpha(x,t) | \leq Ct^{-n/(2\alpha)}. \tag{39}$$

Substituting (39) into (38), we obtain

$$| u(x,t) | \leq Ct^{-n/(2\alpha)} \| u_0 \|_{L^1}. \tag{40}$$

Taking the supremum over $x \in \mathbb{R}^n$in (40), we conclude

$$\| u(\cdot,t) \|_{L^\infty} \leq Ct^{-n/(2\alpha)} \| u_0 \|_{L^1}, \tag{41}$$

which proves (36).

These results show that although both models describe diffusion phenomena, their decay rates are different. In particular, the fractional diffusion equation spreads more slowly in time but produces heavier spatial tails than the classical heat equation.

Comparison Between Classical and Fractional Diffusion

In this section, we compare the analytical properties of the classical heat equation and the fractional diffusion equation. The comparison is based on the Fourier representations and

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the analytical properties discussed in the previous parts.

1. Limit Behavior as $\alpha \to 1$

The fractional diffusion equation can be viewed as a generalization of the classical heat equation. The following result shows that the classical heat equation is recovered as a limiting case.

Theorem 7. Let $u_\alpha(x,t)$ be the solution of the fractional diffusion equation with Fourier representation (13). Then as $\alpha \to 1$, the solution $u_\alpha(x,t)$converges to the classical heat solution $u(x,t)$ given by (9).
Proof. From (13), the Fourier representation of the fractional diffusion solution is

$$\widehat{u_\alpha}(\xi,t) = e^{-|\xi|^{2\alpha}t}\hat{u}_0(\xi). \tag{42}$$

Taking the limit as $\alpha \to 1$, we obtain

$$\lim_{\alpha\to 1} e^{-|\xi|^{2\alpha}t} = e^{-|\xi|^2 t}. \tag{43}$$

Therefore

$$\lim_{\alpha\to 1} \widehat{u_\alpha}(\xi,t) = e^{-|\xi|^2 t}\hat{u}_0(\xi), \tag{44}$$

which coincides with the Fourier representation of the classical heat equation (9). Applying the inverse FT (1) yields the desired result.

2. Comparison of Decay Rates

We now compare the decay rates derived in Section 4.

Theorem 8. Let $u(x,t)$and $u_\alpha(x,t)$be the solutions of the classical heat equation and fractional diffusion equation, respectively. Then the following decay estimates hold

$$\| u(\cdot,t) \|_{L^\infty} \leq Ct^{-n/2} \| u_0 \|_{L^1}, \tag{45}$$

and

$$\| u_\alpha(\cdot,t) \|_{L^\infty} \leq Ct^{-n/(2\alpha)} \| u_0 \|_{L^1}. \tag{46}$$

Proof. The first estimate follows directly from Lemma (29), while the second follows from Lemma (36). Since $0 < \alpha < 1$, we have

$$\frac{n}{2\alpha} > \frac{n}{2}. \tag{47}$$

Thus the fractional diffusion equation exhibits a slower decay rate compared with the classical heat equation.

3. Comparison of Diffusion Behavior

Another important difference between the two models can be observed in their Fourier multipliers.

Lemma 5. The Fourier multipliers associated with the classical and fractional diffusion operators are

$$| \xi |^2 \text{ and } | \xi |^{2\alpha}, \tag{48}$$

respectively.
Proof. This result is a direct consequence of the FT of the Laplacian (3) together with the characterization of the fractional Laplacian (4).

The difference in the Fourier multipliers leads to fundamentally different damping

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mechanisms in the frequency domain. In particular, since $0 < \alpha < 1$, we have

$$|\xi|^{2\alpha} \ll |\xi|^2 \text{ for large } |\xi|, \tag{49}$$

which implies that the exponential decay

$$e^{-|\xi|^{2\alpha}t},$$

is significantly weaker than

$$e^{-|\xi|^{2}t}.$$

As a consequence, high-frequency components are damped much faster in the classical heat equation, resulting in stronger smoothing effects. In contrast, the fractional diffusion equation preserves more high-frequency content, which leads to slower regularization and the appearance of heavy-tailed spatial behavior. This observation provides a clear analytical explanation of the transition from local Gaussian diffusion to nonlocal anomalous diffusion governed by fractional operators.

CONCLUSION

This paper investigates the classical heat equation and the fractional diffusion equation using FT methods. By applying the FT, both equations were reduced to ordinary differential equations in the frequency domain, leading to explicit Fourier representations of their solutions. The analysis shows that the classical heat equation exhibits stronger smoothing and faster decay compared with the fractional diffusion equation, while the fractional model produces heavier spatial tails due to the nonlocal nature of the fractional Laplacian. Moreover, the classical heat equation emerges from the fractional model as $\alpha \to 1$. Future research may consider further analytical properties of fractional diffusion equations, including extensions to nonlinear models or other types of fractional operators.

AUTHOR CONTRIBUTIONS

The first author contributed to conceptualization, methodology, formal analysis, investigation, and writing—original draft. The second author contributed to the algebraic analysis and validation of the results. Both authors were involved in writing—review and editing. Supervision was conducted by the first author.

CONFLICT OF INTEREST

The authors declare no conflicts of interest.

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